Finally, we consider an example of the postbuckling behavior and imperfection sensitivity of an annular plate. In Fig. 3, average stress-end shortening and average stress-buckling deflection curves are shown for a simply supported annular plate with $\nu = \frac{1}{3}$, $n = 12$, $R_i/R_0 = 0.4$, $\rho = 1.28$, and $\sigma_c/\sigma_y = 0.61$. The bifurcation stress for a similar plate with purely elastic material response ($n = 1$) is $\sigma_c/\sigma_y = 0.82$.

The imperfection is taken to be proportional to the buckling mode obtained from the bifurcation analysis. The buckling mode $\hat{w}(r)$, is normalized so that $\hat{w}(R_i) = 1$ and the imperfection amplitude is denoted by $\hat{\xi}$. The buckling deflection is measured by ξ where

$$\xi = w(R_i) - \hat{\xi} \tag{5}$$

here $w(R_i)$ is the lateral deflection at the plate's inner edge. Furthermore, the results given here for a "perfect" plate are actually the results of a calculation using a small imperfection, $\hat{\xi} = 10^{-4}t$.

In Fig. 3a, the applied edge displacement U is normalized by U_y, that value of U at which initial yielding occurs in the perfect plate. Also shown is the average stress-applied displacement curve for the prebuckling solution. Note that for a perfect plate, the curve continues to rise after bifurcation and differs little from the corresponding curve of the prebuckling solution until the maximum load has been reached.

At bifurcation the elastic-plastic boundary of the perfect plate has reached $r = 0.53\ R_0$, and unloading begins at $r = R_i = 0.4\ R_0$ and $z = t/2$. At the maximum load point, this side ($z = t/2$) of the plate has completely unloaded while on the opposite side of the plate the plastic zone extends to $r = 0.79\ R_0$. This side continues to load plastically until the entire side has yielded. During the latter stages of deformation shown in Fig. 3, a region near $r = R_i = 0.4$, $z = t/2$ has yielded in tension.

This example was chosen because bifurcation occurs at the "knee" of the average stress-applied displacement curve of the prebifurcation solution. The postbuckling behavior of the perfect plate is qualitatively similar to that of a circular plate for which the bifurcation stress is near σ_y.[7] The imperfection sensitivity of the annular plate is not as great as that of the circular plate, but is still significant. For an imperfection amplitude $\hat{\xi} = 0.06\ t$ the maximum load is 82% of the bifurcation load.

Thus, annular plates can be imperfection sensitive when bifurcation of the perfect plate occurs from a partly plastic state.

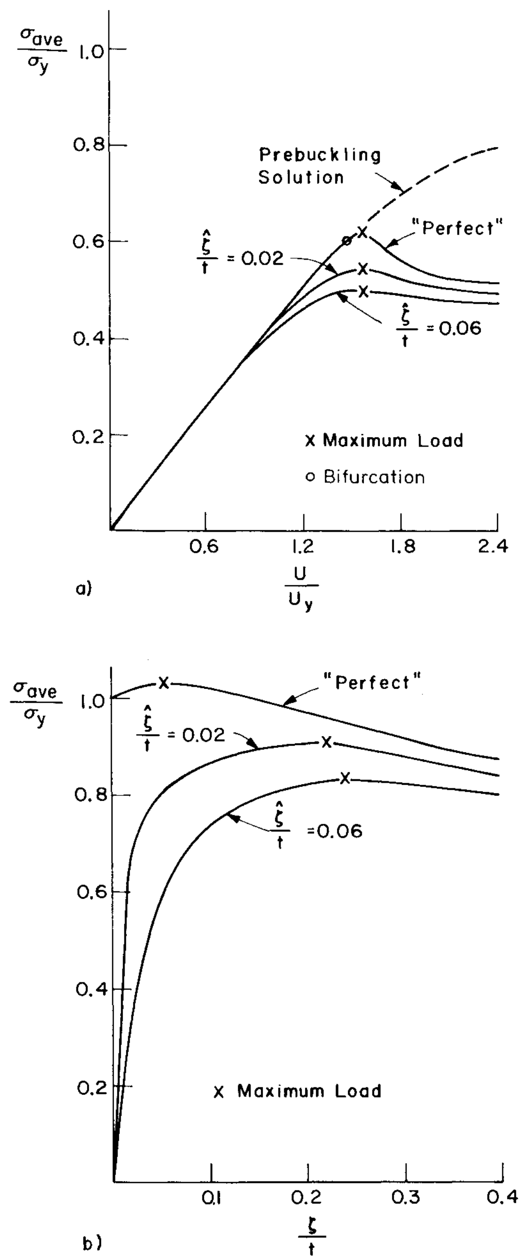

Fig. 3 Postbuckling curves for a simply supported annular plate ($\nu = \frac{1}{3}$, $n = 12$, $R_i/R_0 = 0.4$, $\rho = 1.28$, $\sigma_c/\sigma_y = 0.61$). a) Average stress-applied displacement. b) Average stress-buckling deflection.

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Value of Mach Angle for a Given Prandtl-Meyer Angle

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THIS Note presents a numerical method for determining the Mach angle μ for a given value of the Prandtl-Meyer angle ν.

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The Prandtl-Meyer angle may be written[1]

$$\nu = (1/g)\tan^{-1}[g(M^2-1)^{1/2}] - \tan^{-1}(M^2-1)^{1/2} \quad (1)$$

where M is the Mach Number, $1 \leqq M \leqq \infty$,

$$g = [(\gamma-1)/(\gamma+1)]^{1/2} \quad (2)$$

and γ is the specific heat ratio. Let

$$\mu + \varepsilon = \pi/2 \quad (3)$$

where μ is the Mach angle defined by

$$\sin\mu = 1/M \quad (4)$$

From Eqs. (3) and (4)

$$(M^2-1)^{1/2} = \tan\varepsilon \quad (5)$$

Using Eq. (5), Eq. (1) may be written

$$\tan g(\nu+\varepsilon) = g\tan\varepsilon \quad (6)$$

In Eq. (6), the relation between ν and ε is still implicit. For $\gamma = \frac{5}{3}$, $g = \frac{1}{2}$ and Eq. (6) can be reduced to the explicit form

$$\tan^3(\varepsilon/2) = \tan(\nu/2) \quad (7)$$

Thus, using Eqs. (3) and (7), the simple relation may be written when $\gamma = \frac{5}{3}$

$$\mu = \pi/2 - 2\tan^{-1}[(\tan\nu/2)^{1/3}] \quad (8)$$

This analytical result was first given by Probstein[2] in terms of M, rather than μ, along with a more complicated expression when $\gamma = \frac{5}{4}$. There are no analytical expressions, $\mu = \mu(\nu, \gamma)$, for other values of γ, consequently, μ must be determined from tables, or numerically, in all other cases.

Equation (6) may be written in a form for iteration which always converges to the correct value of ε

$$\varepsilon_{i+1} = \tan^{-1}[(1/g)\tan g(\nu+\varepsilon_i)] \quad (9)$$

for $0 \leqq \varepsilon_i \leqq \pi/2$ and $0 \leqq \nu \leqq \nu_{max}$. The value ν_{max} is given by Eq. (1) for M infinite

$$\nu_{max} = (\pi/2)\{[(\gamma+1)/(\gamma-1)]^{1/2} - 1\} \quad (10)$$

One may start the iteration of Eq. (9) with an approximation for ε_1 given by Eq. (7)

$$\varepsilon_1 = 2\tan^{-1}[(\tan\nu/2)^{1/3}] \quad (11)$$

This approximation fails for large values of ν since it may give a value $\varepsilon_1 > \pi/2$. When this occurs, set $\varepsilon_1 = \pi/2$ to start the iteration with Eq. (9).

While Eq. (9) will always converge to the correct value of ε, its convergence is slow and, after two or three iterations, should be replaced by a method utilizing the derivatives. In effect, Eq. (9) is used to find an approximation of ε which lies in the range of convergence of a faster method.

One method which provides rapid convergence may be written from Eq. (6) by using a Taylor's series expansion for the separate sides about ε_i and solving for $\Delta\varepsilon$. Let

$$f_{1_i} = \tan g(\nu+\varepsilon_i) \quad (12)$$

$$f_{2_i} = g\tan\varepsilon_i \quad (13)$$

Then

$$\Delta\varepsilon_{j+1} = \left(f_{1_i} - f_{2_i} + \frac{\Delta\varepsilon_j^2}{2!}(f_{1_i}'' - f_{2_i}'') + \frac{\Delta\varepsilon_j^3}{3!}(f_{1_i}''' - f_{2_i}''') + \cdots\right) \Big/ (f_{2_i}' - f_{1_i}') \quad (14)$$

where $\Delta\varepsilon_1 = 0$ starts the iteration in Eq. (14). The advantage of this form lies in the fact that the derivatives may be written in terms of the values of f_{1_i} and f_{2_i}, i.e., f_{1_i}' and f_{2_i}' involve the $\sec^2 x$, which can be written as $1+\tan^2 x$, etc. Thus the derivatives may be included with only two numerical evaluations by the computer of the tangent functions in Eqs. (12) and (13). Depending on the number of derivatives included, two or three iterations with Eq. (14) should be sufficient. Then, of course

$$\varepsilon_{i+1} = \varepsilon_i + \Delta\varepsilon_{j+1} \quad (15)$$

is used to evaluate $f_{1_{i+1}}$ and $f_{2_{i+1}}$, etc. For $\gamma = 1.4$ and including the fourth derivative, the above procedure will provide twelve place accuracy for ε in two iterations. With the required value for ε, μ can be written using Eq. (3).

The foregoing result will be useful when studying flowfields by the method of characteristics, where at each computation point the value of μ must be determined for the resulting value of ν.

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On a Class of Fully Stressed Trusses

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Introduction

PREVIOUS investigations by Dayaratnam and Patnaik[1,2] have established conditions necessary for the existence of fully stressed indeterminate trusses. It has been shown that for trusses subjected to single load conditions, only certain geometric configurations are capable of compatible displacements in a fully-stressed state. The examples presented in Refs. 1 and 2 to illustrate this fact are special cases of a general class of fully stressed trusses whose geometry is based on chords of circles. The existence of a class of structures of this type is also referred to in Ref. 3. The purpose of this Note is to describe some interesting characteristics of this class of "circle-chord" trusses which have not been previously reported.

Circle-Chord Trusses

Consider an indeterminate elastic truss composed of a single material with a specified limiting stress σ^* and subjected to a single load condition. Denote the limiting strain by ε^*, where $\varepsilon^* = \sigma^*/E$, and E = Young's modulus. The truss is defined as fully strained, and hence fully stressed, if the strain ε_i $(i = 1, \ldots, n)$ in each of the n members is a constant, ε^*. A fully stressed truss can be obtained only if there exists a set of nodal displacements compatible with the required strain field. The possibility of satisfying this condition depends solely on the configuration of the truss, and is independent of the cross-sectional areas A_i and of the external loads.[1–3] If such a set of displacements exists, and if it is also possible to specify a set of areas A_i for which internal forces equilibrate the external loads, then the desired fully stressed design is obtained. The particular class of fully stressed indeterminate trusses described herein can be made to equilibrate a wide range of loads applied at one node.

In order to establish the general nature of the trusses under investigation, consider the typical component of length l_{AB} in Fig. 1. Let δ_A and δ_B be specified magnitudes of displacements of nodes A and B, with directions given by α_{AB} and β_{AB}, respectively. It follows that component AB will be fully stressed only if its length, l_{AB}, satisfies the relation

$$l_{AB} = (\delta_A\cos\alpha_{AB} - \delta_B\cos\beta_{AB})/\varepsilon^* \quad (1)$$

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